Note on the PROMETHEE net flow computation

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Introduction

This note describes special cases for the computation of the PROMETHEE net flow.
In each case we consider a multicriteria decision problem including a set of n actions:

$$A = \{a_1, a_2, \ldots, a_i, \ldots, a_n\} \tag{1}$$

and k criteria that have to be maximized:

$$f_1, f_2, \ldots, f_j, \ldots, f_k \tag{2}$$

The weights w_j of the criteria are normalized in the following way:

$$\sum_{j=1}^{k} w_j = 1 \tag{3}$$

and the preference functions associated to the criteria are noted as follows:

$$P_j(.,.) \tag{4}$$

The unicriterion net flow for criterion f_j is then defined as:

$$\phi_j(a) = \frac{1}{n-1} \sum_{b \neq a} \left[P_j(a,b) - P_j(b,a) \right] \tag{5}$$

And the multicriteria net flow is the weighted sum of the unicriterion net flows:

$$\phi(a) = \sum_{j=1}^{k} w_j \phi_j(a) \tag{6}$$

Type 3 – V-shape preference function

Let us consider that a V-shape preference function is associated to criterion f_j and that the preference threshold p_j is such that:

$$p_j \geq \max_{a \in A} f_j(a) - \min_{a \in A} f_j(a) \tag{7}$$

For instance, for evaluations between 0 and 1, p_j should be larger than or equal to 1.
In that case the preference function is linear (for positive deviations) and thus:

$$P_j(a,b) - P_j(b,a) = \frac{1}{p_j} \left[f_j(a) - f_j(b) \right] \tag{8}$$

The unicriterion net flow becomes:

$$\begin{aligned}\phi_j(a) &= \frac{1}{n-1}\sum_{b\neq a}\frac{1}{p_j}\left[f_j(a)-f_j(b)\right] \\ &= \frac{1}{n-1}(n-1)\frac{1}{p_j}f_j(a)-\frac{1}{n-1}\frac{1}{p_j}\sum_{b\neq a}f_j(b) \\ &= \frac{1}{p_j}f_j(a)-\frac{1}{n-1}\frac{1}{p_j}\left[n\overline{f_j}-f_j(a)\right] \\ &= \frac{n}{n-1}\frac{1}{p_j}\left[f_j(a)-\overline{f_j}\right]\end{aligned} \tag{9}$$

where $\overline{f_j}$ is the arithmetic average of the evaluations.

The unicriterion net flow value is thus a linear function of the evaluation.
The multicriteria net flow also is a linear function of a weighted sum of the evaluations:

$$\phi(a)=\frac{n}{n-1}\left[\sum_{j=1}^{k}\frac{w_j}{p_j}f_j(a)-\sum_{j=1}^{k}\frac{w_j}{p_j}\overline{f_j}\right] \tag{10}$$

When all the criteria have the same preference threshold p the multicriteria net flow is a linear function of the weighted sum of the evaluations with weights w_j and PROMETHEE II is equivalent to the weighted sum method:

$$\phi(a)=\frac{n}{n-1}\frac{1}{p}\left[\sum_{j=1}^{k}w_jf_j(a)-\sum_{j=1}^{k}w_j\overline{f_j}\right] \tag{11}$$

Type 1 – Usual criterion

For a usual criterion, the preference function is such that:

$$P_j(a,b)-P_j(b,a)=I_{f_j(a)>f_j(b)}-I_{f_j(a)<f_j(b)} \tag{12}$$

where I are indicator (binary) variables. Thus:

$$\sum_{b\neq a}\left[P_j(a,b)-P_j(b,a)\right]=\left|\left\{b\,|\,f_j(a)>f_j(b)\right\}\right|-\left|\left\{b\,|\,f_j(a)<f_j(b)\right\}\right| \tag{13}$$

and

$$\phi_j(a)=\frac{1}{n-1}\left[\left|\left\{b\,|\,f_j(a)>f_j(b)\right\}\right|-\left|\left\{b\,|\,f_j(a)<f_j(b)\right\}\right|\right] \tag{14}$$

From the definition of the average rank it comes that:

$$\phi_j(a)=\frac{n+1}{n-1}-\frac{2}{n-1}R_j(a) \tag{15}$$

where $R_j(a)$ is the average rank of $f_j(a)$ in the set of the evaluations for criterion f_j.
When all the criteria are associated to usual preference functions, the multicriteria net flow thus becomes:

$$\phi(a) = \frac{n+1}{n-1} - \frac{2}{n-1}\sum_{j=1}^{k} w_j R_j(a) \tag{16}$$

In this case the PROMETHEE II ranking is equivalent to the Borda method.